ON THE AXIALLY-SYMMETRIC DEFORMATIONS OF A SOLID CIRCULAR CYLINDER OF FINITE LENGTH

(OB OSESIMMETRICHNOI DEFORMATSII SPLOSHNOGO KRUGOVOGO TSILINDRA KONECHNOI DLINY)

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In this paper we solve the fundamental mixed problem and the second fundamental problem of the theory of elasticity for the axially symmetric deformations of a circular cylinder of finite length. Two variations of the mixed problem are solved: 1) arbitrary stresses are prescribed on the ends of the cylinder and displacements are prescribed on the lateral surface, and 2) arbitrary displacements are given on the ends of the cylinder and arbitrary stresses are prescribed on the lateral surface. A particular case of the first problem is the bending of a thick circular plate whose lateral surface is rigidly clamped and which is acted upon by a load applied over one of the end surfaces.

The mixed problem for the cylinder was examined by Filon [1]; however, the boundary conditions pertaining to the tangential displacements on the ends of the cylinder were not satisfied. This problem was solved by an approximate method in [2]. Other mixed problems concerning the elastic deformations of a finite length have been investigated in a number of papers [3, 14].

1. Stresses prescribed on the ends of the cylinder and displacements prescribed on the lateral surface. It is required to find functions $u(r, z)$ and $w(r, z)$ which in the interior of the cylinder $0 \leqslant r \leqslant R$, $-l \leqslant z \leqslant l$ satisfy the Lamé differential equations

$$\frac{2(1-\sigma)}{1-2\sigma}\frac{\partial\theta}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = 0$$
$$\frac{2(1-\sigma)}{1-2\sigma}\frac{\partial\theta}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial z\,\partial r} = 0 \qquad (1.1)$$

and which on the surface of the cylinder satisfy the conditions

$$w(R, z) = \chi(z), \quad \tau_{rz}(r, l) = f_1(r), \quad \tau_{rz}(r, -l) = f_2(r) \tag{1.2}$$
$$u(R, z) = \psi(z), \quad \sigma_z(r, l) = \varphi_1(r), \quad \sigma_z(r, -l) = \varphi_2(r) \tag{1.3}$$

Here σ is Poisson's ratio, G is the shear modulus, θ is the dilatation, and $\sigma_z(r, z)$ and $\tau_{rz}(r, z)$ are the stress tensor components, viz.

$$\theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}, \quad \sigma_z = 2G\left(\frac{\partial w}{\partial z} + \frac{\sigma}{1-2\sigma}\theta\right), \quad \tau_{rz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{1.4}$$

The boundary functions $\varphi_1(r)$, $\varphi_2(r)$, $f_1(r)$, and $f_2(r)$ are assumed to admit of a Fourier representation in terms of Bessel functions of the first kind

$$\varphi_i(r) = \sum_{n=1}^{\infty} \varphi_n^{(i)} J_0(\lambda_n r), \quad f_i(r) = \sum_{n=1}^{\infty} f_n^{(i)} J_1(\lambda_n r) \quad (i = 1, 2) \quad (0 \leqslant r \leqslant R) \tag{1.5}$$

where $\lambda_n R = \mu_n$ are the positive roots of the equation $J_0(\mu) = 0$

$$\varphi_n^{(i)} = \frac{2}{R^2 J_1^2(\mu_n)} \int_0^R r\varphi_i(r) J_0(\lambda_n r)\, dr, \quad f_n^{(i)} = \frac{2}{R^2 J_1^2(\mu_n)} \int_0^R r f_i(r) J_1(\lambda_n r)\, dr \tag{1.6}$$

and the functions $\psi(z)$ and $\chi(z)$ are Fourier series

$$\psi(z) = \frac{\psi_0}{2} + \sum_{m=1}^{\infty} \psi_m \cos\frac{m\pi(z-l)}{2l}$$
$$\chi(z) = \sum_{m=1}^{\infty} \chi_m \sin\frac{m\pi(z-l)}{2l} \qquad (-l \leqslant z \leqslant l) \tag{1.7}$$

To solve the problem we introduce the Papkovich-Neuber representation of the solution of the Lamé equations. In the case of axially-symmetric deformations this takes on the following form in cylindrical coordinates

$$u = -\frac{1}{4(1-\sigma)}\frac{\partial}{\partial r}(z\delta_1 + \delta) - \frac{1}{2(1-\sigma)}\left[(4\sigma - 1)\frac{\partial \delta_2}{\partial r} + r\frac{\partial^2 \delta_2}{\partial r^2}\right]$$
$$w = \delta_1 - \frac{1}{4(1-\sigma)}\frac{\partial}{\partial z}(z\delta_1 + \delta) - \frac{1}{2(1-\sigma)}\left[2\frac{\partial \delta_2}{\partial z} + r\frac{\partial^2 \delta_2}{\partial r\, \partial z}\right] \tag{1.8}$$

where δ, δ_1, and δ_2 are arbitrary harmonic functions. Setting in (1.8)

$$\delta_1 = 0, \ \delta = \frac{C_m^{(1)} I_0(k_m r)}{k_m^2 I_0(k_m R)} \cos\frac{m\pi(z-l)}{2l}, \quad \delta_2 = \frac{C_m I_0(k_m r)}{k_m^2 I_0(k_m R)} \cos\frac{m\pi(z-l)}{2l}$$

we obtain the following particular solutions of the Lamé equations (1.1):

$$u_m^{(1)} = -\left\{\frac{1}{2} C_m^{(1)} I_1(k_m r) + C_m [(4\sigma - 2) I_1(k_m r) + \right.$$
$$\left. + k_m r I_0(k_m r)]\right\} \frac{\cos [m\pi(z-l)/2l]}{2(1-\sigma) k_m I_0(k_m R)} \tag{1.9}$$
$$w_m^{(1)} = \left\{\frac{1}{2} C_m^{(1)} I_0(k_m r) + C_m [2I_0(k_m r) + k_m r I_1(k_m r)]\right\} \frac{\sin[m\pi(z-l)/2l]}{2(1-\sigma) k_m I_0(k_m R)}$$

here C_m, $C_m^{(1)}$ are arbitrary constants, and $I_0(k_m r)$, $I_1(k_m r)$ are modified Bessel functions

$$k_m = \frac{m\pi}{2l} \qquad (m = 1, 2, \ldots) \tag{1.10}$$

Further, by setting in (1.8)

$$\delta = \frac{1}{\lambda_n J_1(\lambda_n R) \sinh \lambda_n l} [A_n^{(3)} \tanh \lambda_n l \sinh \lambda_n z + A_n^{(4)} \cosh \lambda_n z] J_0(\lambda_n r)$$
$$\delta_1 = \frac{1}{\lambda_n J_1(\lambda_n R) \sinh \lambda_n l} [A_n^{(1)} \sinh \lambda_n z + A_n^{(2)} \tanh \lambda_n l \cosh \lambda_n z] J_0(\lambda_n r)$$
$$\delta_2 = 0, \qquad \lambda_n R = \mu_n, \qquad J_0(\mu_n) = 0, \qquad (n = 1, 2, \ldots) \tag{1.11}$$

we obtain a second type of particular solution of Equations (1.1)

$$u_n^{(2)} = \frac{1}{4(1-\sigma) J_1(\lambda_n R) \sinh \lambda_n l} [A_n^{(3)} \tanh \lambda_n l \sinh \lambda_n z + A_n^{(4)} \cosh \lambda_n z +$$
$$+ A_n^{(1)} z \sinh \lambda_n z + A_n^{(2)} \tanh \lambda_n l \, z \cosh \lambda_n z] J_1(\lambda_n r)$$
$$w_n^{(2)} = \frac{1}{4(1-\sigma) J_1(\lambda_n R) \sinh \lambda_n l} [- A_n^{(3)} \tanh \lambda_n l \cosh \lambda_n z - A_n^{(4)} \sinh \lambda_n z - \tag{1.12}$$
$$- A_n^{(1)} z \cosh \lambda_n z - A_n^{(2)} \tanh \lambda_n l \, z \sinh \lambda_n z +$$
$$+ \frac{3-4\sigma}{\lambda_n} (A_n^{(1)} \sinh \lambda_n z + A_n^{(2)} \tanh \lambda_n l \cosh \lambda_n z)] J_0(\lambda_n r)$$

where $A_n^{(1)}$, $A_n^{(2)}$, $A_n^{(3)}$, $A_n^{(4)}$ are arbitrary constants.

We assume a series solution of the boundary value problem formulated above

$$u = \frac{a_2 r}{8(1-\sigma)} + \sum_{m=1}^{\infty} u_m^{(1)} + \sum_{n=1}^{\infty} u_n^{(2)}, \qquad w = \sum_{m=1}^{\infty} w_m^{(1)} + \sum_{n=1}^{\infty} w_n^{(2)} \tag{1.13}$$

It is evident that the term containing the arbitrary constant a_2 satisfies Equations (1.1).

The satisfaction of the boundary conditions (1.2), together with the consideration of the expansions (1.5) and (1.7), leads to three identities. Equating the Fourier coefficients of the functions on the left and right sides of these identities, we obtain the following relations for the unknown constants

$$C_m^{(1)} = -2C_m\left[2 + k_m R\,\frac{I_1(k_m R)}{I_0(k_m R)}\right] + 4(1-\sigma)\,k_m \chi_m$$

$$A_n^{(3)} = \frac{A_n^{(2)}}{\lambda_n}\,[1 - 2\sigma - \lambda_n l \tanh \lambda_n l] + \frac{1-\sigma}{G\lambda_n}\,[f_n^{(1)} + f_n^{(2)}]\,J_1(\lambda_n R) \qquad (1.14)$$

$$A_n^{(4)} = \frac{A_n^{(1)}}{\lambda_n}\,[1 - 2\sigma - \lambda_n l \coth \lambda_n l] + \frac{1-\sigma}{G\lambda_n}\,[f_n^{(1)} - f_n^{(2)}]\,J_1(\lambda_n R)$$

If we satisfy the remaining boundary conditions (1.3), taking the expansions (1.5) and (1.7) into account, we obtain three equations

$$\frac{a_2 R}{8(1-\sigma)} + \sum_{n=1}^{\infty} F_1(z, n) - \sum_{m=1}^{\infty} R_m^{(1)} \cos\frac{m\pi(z-l)}{2l} = \frac{\psi_0}{2} + \sum_{m=1}^{\infty} \psi_m \cos\frac{m\pi(z-l)}{2l}$$

$$\frac{\sigma G a_2 (1-\sigma)^{-1}}{2(1-2\sigma)} + \sum_{m=1}^{\infty} F_2(r, m) + \frac{1}{2}\sum_{n=1}^{\infty} [R_n^{(2)} + R_n^{(3)}]\,J_0(\lambda_n r) = \sum_{n=1}^{\infty} \varphi_n^{(1)} J_0(\lambda_n r)$$

$$\frac{\sigma G a_2}{2(1-\sigma)(1-2\sigma)} + \sum_{m=1}^{\infty} (-1)^m F_2(r, m) + \frac{1}{2}\sum_{n=1}^{\infty} [R_n^{(2)} - R_n^{(3)}]\,J_0(\lambda_n r) = \sum_{n=1}^{\infty} \varphi_n^{(2)} J_0(\lambda_n r) \qquad (1.15)$$

in which

$$F_1(z, n) = \frac{1}{4(1-\sigma)\sinh \lambda_n l}\,[A_n^{(3)} \tanh \lambda_n l \sinh \lambda_n z + A_n^{(4)} \cosh \lambda_n z + A_n^{(1)} z \sinh \lambda_n z + A_n^{(2)} \tanh \lambda_n l\; z \cosh \lambda_n z] \qquad (1.16)$$

$$F_2(r, m) = \frac{G}{(1-\sigma)\,I_0(k_m R)}\left\{\frac{1}{2}\,C_m^{(1)} I_0(k_m r) + C_m[2(1+\sigma)\,I_0(k_m r) + k_m r I_1(k_m r)]\right\}$$

$$R_m^{(1)} = \frac{1}{2(1-\sigma)\,k_m I_0(k_m R)}\left\{\frac{1}{2}\,C_m^{(1)} I_1(k_m R) + C_m[(4\sigma - 2)\,I_1(k_m R) + k_m R I_0(k_m R)]\right\} \qquad (1.17)$$

$$R_n^{(2)} = \frac{1}{J_1(\lambda_n R)}\left[-\frac{G\lambda_n}{1-\sigma} A_n^{(4)} \coth\lambda_n l + 2GA_n^{(1)} \coth\lambda_n l - \frac{\lambda_n G}{1-\sigma} A_n^{(1)} l\right]$$

$$R_n^{(3)} = \frac{1}{J_1(\lambda_n R)}\left[-\frac{G\lambda_n}{1-\sigma} A_n^{(3)} \tanh\lambda_n l + 2GA_n^{(2)} \tanh\lambda_n l - \frac{\lambda_n G}{1-\sigma} A_n^{(2)} l\right]$$

With the help of Formulas (1.14), the expressions for the quantities $R_m^{(1)}$, $R_n^{(2)}$ and $R_n^{(3)}$ are easily transformed into the form

$$\begin{aligned} R_m^{(1)} &= -\frac{C_m L_m}{2(1-\sigma)k_m} + \frac{\chi_m I_1(k_m R)}{I_0(k_m R)} \\ R_n^{(2)} &= \frac{GA_n^{(1)} L_n^{(1)}}{(1-\sigma) J_1(\lambda_n R)} + [f_n^{(2)} - f_n^{(1)}] \coth\lambda_n l \\ R_n^{(3)} &= \frac{GA_n^{(2)} L_n^{(2)}}{(1-\sigma) J_1(\lambda_n R)} - [f_n^{(1)} + f_n^{(2)}] \tanh\lambda_n l \end{aligned} \tag{1.18}$$

where

$$L_n^{(1)} = \coth\lambda_n l + \frac{\lambda_n l}{\sinh^2\lambda_n l}, \qquad L_n^{(2)} = \tanh\lambda_n l - \frac{\lambda_n l}{\cosh^2\lambda_n l} \tag{1.19}$$

$$L_m = (4-4\sigma)\frac{I_1(k_m R)}{I_0(k_m R)} - k_m R\left[1 - \frac{I_1^2(k_m R)}{I_0^2(k_m R)}\right] \tag{1.20}$$

In order to equate the Fourier coefficients of the functions on the left and right sides of Equations (1.15) and thereby to find relations for the unknown constants C_m, $A_n^{(1)}$, $A_n^{(2)}$ and a_2, we expand the functions (1.16) and (1.17) in Fourier series

$$\begin{aligned} F_1(z, n) &= \frac{1}{2} F_0^{(1)}(n) + \sum_{m=1}^{\infty} F_m^{(1)}(n) \cos\frac{m\pi(z-l)}{2l} \qquad (-l \leqslant z \leqslant l) \\ F_2(r, m) &= \sum_{n=1}^{\infty} F_n^{(2)}(m) J_0(\lambda_n r), \qquad J_0(\lambda_n R) = 0 \qquad (0 \leqslant r < R) \end{aligned} \tag{1.21}$$

It is not difficult to calculate the Fourier coefficients and thereby to obtain the expressions

$$\begin{aligned} F_0^{(1)}(n) &= -\frac{\sigma A_n^{(1)}}{l(1-\sigma)\lambda_n^2} + \frac{J_1(\lambda_n R)[f_n^{(1)} - f_n^{(2)}]}{2lG\lambda_n^2} \\ F_m^{(1)}(n) &= \frac{A_n^{(2)} H_{mn}^{(1)}}{2(1-\sigma)k_m} + \frac{[f_n^{(1)} + f_n^{(2)}] J_1(\lambda_n R)}{2lG(k_m^2 + \lambda_n^2)} \qquad (m = 1, 3, \ldots) \\ F_m^{(1)}(n) &= \frac{A_n^{(1)} H_{mn}^{(1)}}{2(1-\sigma)k_m} + \frac{[f_n^{(1)} - f_n^{(2)}] J_1(\lambda_n R)}{2lG(k_m^2 + \lambda_n^2)} \qquad (m = 2, 4, \ldots) \end{aligned} \tag{1.22}$$

$$F_n^{(2)}(m) = \frac{GC_m H_{mn}^{(2)}}{2(1-\sigma) J_1(\lambda_n R)} + \frac{4G\chi_m k_m \lambda_n}{R(\lambda_n^2 + k_m^2) J_1(\lambda_n R)}$$

Here

$$H_{mn}^{(1)} = \frac{2k_m [k_m^2 - \sigma(\lambda_n^2 + k_m^2)]}{l(k_m^2 + \lambda_n^2)^2}, \qquad H_{mn}^{(2)} = \frac{8\lambda_n [\sigma(k_m^2 + \lambda_n^2) - k_m^2]}{R[\lambda_n^2 + k_m^2]^2} \quad (1.23)$$

Now, setting the series (1.21) and the expansion of unity

$$1 = \frac{2}{R} \sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n R)} J_0(\lambda_n r) \qquad (0 \leqslant r < R)$$

into Equations (1.15) and equating Fourier coefficients, we obtain four relationships. Transforming these relationships with the help of Formulas (1.18) and (1.22), we find the constants a_2, C_m, $A_n^{(1)}$, and $A_n^{(2)}$

$$a_2 = \frac{4\sigma}{Rl} \sum_{n=1}^{\infty} \frac{A_n^{(1)}}{\lambda_n^2} - \frac{2(1-\sigma)}{GlR} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} J_1(\lambda_n R)[f_n^{(1)} - f_n^{(2)}] + \frac{4(1-\sigma)\psi_0}{R} \quad (1.24)$$

$$C_m = -\frac{1}{L_m} \sum_{n=1}^{\infty} A_n^{(2)} H_{mn}^{(1)} + \xi_m^{(2)} \qquad (m = 1, 3, \ldots) \quad (1.25)$$

$$C_m = -\frac{1}{L_m} \sum_{n=1}^{\infty} A_n^{(1)} H_{mn}^{(1)} + \xi_m^{(1)} \qquad (m = 2, 4, \ldots) \quad (1.26)$$

$$A_n^{(1)} = -\frac{1}{L_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} C_m H_{mn}^{(2)} - \frac{8\sigma^2}{R^2 l(1-2\sigma)\lambda_n L_n^{(1)}} \sum_{s=1}^{\infty} \frac{A_s^{(1)}}{\lambda_s^2} + \eta_n^{(1)}$$

$$(n = 1, 2, \ldots) \quad (1.27)$$

$$A_n^{(2)} = -\frac{1}{L_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} C_m H_{mn}^{(2)} + \eta_n^{(2)} \qquad (n = 1, 2, \ldots) \quad (1.28)$$

where

$$\xi_m^{(2)} = \frac{2(1-\sigma) k_m}{L_m} \left[\frac{\chi_m I_1(k_m R)}{I_0(k_m R)} + \psi_m - \sum_{n=1}^{\infty} \frac{[f_n^{(1)} + f_n^{(2)}] J_1(\lambda_n R)}{2Gl(\lambda_n^2 + k_m^2)} \right]$$

$$\xi_m^{(1)} = \frac{2(1-\sigma) k_m}{L_m} \left[\frac{\chi_m I_1(k_m R)}{I_0(k_m R)} + \psi_m - \sum_{n=1}^{\infty} \frac{[f_n^{(1)} - f_n^{(2)}] J_1(\lambda_n R)}{2Gl(k_m^2 + \lambda_n^2)} \right]$$

$$\eta_n^{(1)} = \frac{(1-\sigma) J_1(\lambda_n R)}{GL_n^{(1)}} [(f_n^{(1)} - f_n^{(2)}) \coth \lambda_n l + \varphi_n^{(1)} + \varphi_n^{(2)}] -$$

$$- \frac{8(1-\sigma)}{RL_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} \frac{\chi_m k_m \lambda_n}{(\lambda_n^2 + k_m^2)} -$$

$$- \frac{2\sigma}{R(1-2\sigma)\lambda_n L_n^{(1)}} \left\{ \frac{4(1-\sigma)\psi_0}{R} - \frac{2(1-\sigma)}{lRG} \sum_{s=1}^{\infty} \frac{J_1(\lambda_s R)}{\lambda_s^2} [f_s^{(1)} - f_s^{(2)}] \right\}$$

$$\eta_n^{(2)} = \frac{(1-\sigma) J_1(\lambda_n R)}{GL_n^{(2)}} [(f_n^{(1)} + f_n^{(2)}) \tanh \lambda_n l + \varphi_n^{(1)} - \varphi_n^{(2)}] -$$

$$- \frac{8(1-\sigma)}{RL_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} \frac{\chi_m k_m \lambda_n}{\lambda_n^2 + k_m^2} \qquad (1.29)$$

From Equations (1.25) to (1.28) we obtain two infinite systems of infinite algebraic equations for the unknown constants $A_n^{(1)}$ and $A_n^{(2)}$

$$A_n^{(1)} = \frac{1}{L_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} \sum_{s=1}^{\infty} \frac{1}{L_m} H_{ms}^{(1)} H_{mn}^{(2)} A_s^{(1)} -$$

$$- \frac{8\sigma^2}{R^2 l (1-2\sigma) \lambda_n L_n^{(1)}} \sum_{s=1}^{\infty} \frac{A_s^{(1)}}{\lambda_s^2} + \delta_n^{(1)} \qquad (n = 1, 2, \ldots) \quad (1.30)$$

$$A_n^{(2)} = \frac{1}{L_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} \sum_{s=1}^{\infty} \frac{1}{L_m} H_{ms}^{(1)} H_{mn}^{(2)} A_s^{(2)} + \delta_n^{(2)} \qquad (n = 1, 2, \ldots) \quad (1.31)$$

Here

$$\delta_n^{(1)} = - \frac{1}{L_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} \xi_m^{(1)} H_{mn}^{(2)} + \eta_n^{(1)}$$

$$\delta_n^{(2)} = - \frac{1}{L_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} \xi_m^{(2)} H_{mn}^{(2)} + \eta_n^{(2)} \qquad (1.32)$$

Thus, the boundary conditions have been satisfied, and the series (1.13) represent a solution of the boundary value problem that has been posed above. The constants a_2, $A_n^{(3)}$, $A_n^{(4)}$, C_m, and $C_m^{(1)}$, entering into the series (1.13), are uniquely expressed in terms of the constants $A_n^{(1)}$, $A_n^{(2)}$, and the Fourier coefficients of the boundary functions by means of Equations (1.24) to (1.26) and (1.14). The constants $A_n^{(1)}$ and $A_n^{(2)}$ are found from the infinite systems (1.30) and (1.31). If there exists a unique bounded solution of the infinite systems (1.30) and (1.31), then the series (1.13), which gives the solution of the problem, converges uniformly in the interior of the cylinder $-l \leqslant z \leqslant l$,

$0 \leqslant r \leqslant R$ and there admits of termwise double differentiation.

2. Investigation of the infinite systems. Before entering into the investigation of the infinite systems (1.30) and (1.31), we establish some identities and inequalities that contain modified Bessel functions. We expand the functions r and $I_1(kr)$ in the interval $0 \leqslant r \leqslant R$ in a Fourier-Dini series

$$r = \sum_{n=1}^{\infty} f_n^{(0)} J_1(\lambda_n r), \qquad I_1(kr) = \sum_{n=1}^{\infty} f_n J_1(\lambda_n r) \tag{2.1}$$

here $\lambda_n R = \mu_n$ are the positive roots of the equation $J_0(\mu) = 0$. Calculating the Fourier coefficients according to the second Formula (1.6), we find

$$f_n^{(0)} = \frac{4}{R\lambda_n^2 J_1(\lambda_n R)}, \qquad f_n = \frac{2kI_0(kR)}{RJ_1(\lambda_n R)(\lambda_n^2 + k^2)}$$

Since the conditions of uniform convergence are fulfilled [15], the series (2.1) converges uniformly on the segment $a \leqslant r \leqslant R$, where $0 < a < R$. Therefore, setting $r = R$ in (2.1) and substituting in the values of $f_n^{(0)}$ and f_n, we obtain two identities

$$\frac{R^2}{4} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \tag{2.2}$$

$$\frac{2}{R} \sum_{n=1}^{\infty} \frac{k}{k^2 + \lambda_n^2} = \frac{I_1(kR)}{I_0(kR)}, \qquad J_0(\lambda_n R) = 0 \tag{2.3}$$

Differentiating (2.3) with respect to the parameter k, we arrive at the identity

$$\frac{4}{R} \sum_{n=1}^{\infty} \frac{k^3}{(k^2 + \lambda_n^2)^2} = 2\frac{I_1(kR)}{I_0(kR)} - kR\left[1 - \frac{I_1^2(kR)}{I_0^2(kR)}\right], \qquad J_0(\lambda_n R) = 0 \tag{2.4}$$

Hence as a consequence of Equations (2.3) and (2.4) we find

$$\frac{4}{R} \sum_{n=1}^{\infty} \frac{k\lambda_n^2}{(\lambda_n^2 + k^2)^2} = kR\left[1 - \frac{I_1^2(kR)}{I_0^2(kR)}\right], \qquad J_0(\lambda_n R) = 0 \tag{2.5}$$

Further, we record the known identities [16]

$$\frac{2}{R} \sum_{n=1}^{\infty} \frac{k}{k^2 + \alpha_n^2} = \frac{I_0(kR)}{I_1(kR)} - \frac{2}{kR} \tag{2.6}$$

$$\frac{4}{R}\sum_{n=1}^{\infty}\frac{k^3}{(k^2+\alpha_n^2)^2}=kR\left[\frac{I_0^2(kR)}{I_1^2(kR)}-1\right]-\frac{4}{kR} \tag{2.7}$$

$$\frac{4}{R}\sum_{n=1}^{\infty}\frac{k\alpha_n^2}{(k^2+\alpha_n^2)^2}=2\frac{I_0(kR)}{I_1(kR)}-kR\left[\frac{I_0^2(kR)}{I_1^2(kR)}-1\right] \tag{2.8}$$

Here $\alpha_n R = \gamma_n$ are the positive roots of the equation $J_1(\gamma) = 0$. From the interlacing of the roots of the Bessel functions $J_0(x)$ and $J_1(x)$ it follows that [15]

$$\lambda_n < \alpha_n, \qquad \lambda_{n+1} > \alpha_n \qquad (n = 1, 2, \ldots) \tag{2.9}$$

Now, from the identities (2.3) and (2.6) and the first inequality (2.9) we have

$$\frac{I_0(kR)}{I_1(kR)}-\frac{2}{kR}<\frac{I_1(kR)}{I_0(kR)}, \qquad kR\left[\frac{I_0(kR)}{I_1(kR)}-\frac{I_1(kR)}{I_0(kR)}\right]<2 \tag{2.10}$$

Since $\lambda_{n+1} > \alpha_n$, it follows that

$$\frac{4}{R}\sum_{n=1}^{\infty}\frac{k^3}{(k^2+\lambda_n^2)^2}-\frac{4}{R}\frac{k^3}{(k^2+\lambda_1^2)^2}<\frac{4}{R}\sum_{n=1}^{\infty}\frac{k^3}{(k^2+\alpha_n^2)^2}$$

Hence

$$\frac{4}{R}\sum_{n=1}^{\infty}\frac{k^3}{(k^2+\lambda_n^2)^2}<\frac{4}{R}\sum_{n=1}^{\infty}\frac{k^3}{(k^2+\alpha_n^2)^2}+\frac{4}{kR}$$

or by virtue of (2.4) and (2.7)

$$2\frac{I_1(kR)}{I_0(kR)}<kR\left[1-\frac{I_1^2(kR)}{I_0^2(kR)}\right]+kR\left[\frac{I_0^2(kR)}{I_1^2(kR)}-1\right]$$

Since

$$kR\left[1-\frac{I_1^2(kR)}{I_0^2(kR)}\right]<kR\left[\frac{I_0^2(kR)}{I_1^2(kR)}-1\right]$$

then from the last inequality and (2.10) we obtain

$$\frac{I_0(kR)}{I_1(kR)}<kR\left[\frac{I_0^2(kR)}{I_1^2(kR)}-1\right]+\frac{2}{kR} \tag{2.11}$$

In the investigation of the infinite systems (1.30) and (1.31) we limit ourselves to Poisson's ratios σ that vary over the interval $0 < \sigma < 1/3$. We denote sums of the moduli of coefficients of the systems

(1.30) and (1.31) by $T_n^{(1)}$ and $T_n^{(2)}$ respectively. We bound them from above. It is seen that

$$T_n^{(1)} \leqslant \frac{1}{L_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} |H_{mn}^{(2)}| \Gamma_m + T_n^{(0)}, \quad T_n^{(2)} \leqslant \frac{1}{L_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} |H_{mn}^{(2)}| \Gamma_m \quad (2.12)$$

$$(n = 1, 2, \ldots)$$

where

$$\Gamma_m = \sum_{s=1}^{\infty} \frac{1}{L_m} |H_{ms}^{(1)}|, \qquad T_n^{(0)} = \frac{8\sigma^2}{R^2 l (1-\sigma) \lambda_n L_n^{(1)}} \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} \quad (2.13)$$

and

$$L_m > 0, \qquad L_n^{(1)} > 0, \qquad L_n^{(2)} = \frac{\sinh 2\lambda_n a - 2\lambda_n a}{2\cosh^2 \lambda_n a} > 0$$

$$|H_{ms}^{(1)}| = \frac{2k_m |k_m^2 - \sigma(\lambda_s^2 + k_m^2)|}{l [k_m^2 + \lambda_s^2]^2} \leqslant \frac{R}{2l} \frac{4}{R} \frac{k_m [(1-\sigma) k_m^2 + \sigma\lambda_s^2]}{(k_m^2 + \lambda_s^2)^2} \quad (2.14)$$

which follows from the identity (2.4) and formulas (1.20), (1.19), and (1.23). Applying (2.14), (2.4), (2.5) and (1.20) we find

$$\Gamma_m \leqslant \frac{R}{2lL_m} \left\{ (1-\sigma) \left[2\frac{I_1(k_m R)}{I_0(k_m R)} - k_m R \left(1 - \frac{I_1^2(k_m R)}{I_0^2(k_m R)} \right) \right] + \right.$$
$$\left. + \sigma k_m R \left[1 - \frac{I_1^2(k_m R)}{I_0^2(k_m R)} \right] \right\} =$$
$$= \frac{R}{2l} \frac{2 - 2\sigma - (1-2\sigma) k_m R [I_0(k_m R)/I_1(k_m R) - I_1(k_m R)/I_0(k_m R)]}{4 - 4\sigma - k_m R [I_0(k_m R)/I_1(k_m R) - I_1(k_m R)/I_0(k_m R)]}$$

or

$$\Gamma_m \leqslant \frac{R}{2l} f(t_m, \sigma) \qquad (m = 1, 2, 3, \ldots) \quad (2.15)$$

Here

$$f(t_m, \sigma) = \frac{2 - 2\sigma - (1-2\sigma) t_m}{4 - 4\sigma - t_m}, \qquad t_m = k_m R \left[\frac{I_0(k_m R)}{I_1(k_m R)} - \frac{I_1(k_m R)}{I_0(k_m R)} \right]$$

By virtue of the inequality (2.10), the argument t_m varies over the interval $0 < t_m < 2$. Let us determine the largest value of the function $f(t_m, \sigma)$ on the interval $0 \leqslant t_m \leqslant 2$ for various values of σ. The derivative

$$\frac{\partial f}{\partial t_m} = \frac{2(5\sigma - 4\sigma^2 - 1)}{(4 - 4\sigma - t_m)^2}$$

does not vanish on the interval $0 \leqslant t_m \leqslant 2$. Since the trinomial $5\sigma - 4\sigma^2 - 1$ has the roots $\sigma_0 = 1/4$ and $\sigma_1 = 1$, we have

$$\partial f / \partial t_m < 0, \quad (0<\sigma<1/4); \qquad \partial f / \partial t_m > 0, \quad (1/4<\sigma\leqslant 1/2)$$

Therefore it follows that

$$\begin{aligned} f(t_m, \sigma) &\leqslant f(0, \sigma) = 1 - 1/2 & (0<\sigma\leqslant 1/4) \\ f(t_m, \sigma) &\leqslant f(2, \sigma) = 1-(1-3\sigma)/(1-2\sigma) & (1/4<\sigma<1/2) \end{aligned} \tag{2.16}$$

We set

$$2\theta_1 = \begin{cases} 1/2 & (0<\sigma\leqslant 1/4) \\ (1-3\sigma)/(1-2\sigma) & (1/4<\sigma<1/2) \end{cases} \tag{2.17}$$

whereby

$$\theta_1 > 0 \qquad (0<\sigma<1/3) \tag{2.18}$$

Hence by virtue of (2.16) and (2.17) the inequality, (2.15) can be rewritten as

$$\Gamma_m \leqslant \frac{R}{2l}(1-2\theta_1) \qquad (m=1, 2, \ldots)$$

From which the inequalities (2.12) can be rewritten as

$$T_n^{(1)} \leqslant \frac{R(1-2\theta_1)}{2lL_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} |H_{mn}^{(2)}| + T_n^{(0)} \quad (n=1, 2, \ldots) \tag{2.19}$$

$$T_n^{(2)} \leqslant \frac{R(1-2\theta_1)}{2lL_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} |H_{mn}^{(2)}| \qquad (n=1, 2, \ldots) \tag{2.20}$$

where

$$|H_{mn}^{(2)}| = \frac{8\lambda_n |\sigma[(m\pi/2l)^2+\lambda_n^2]-(m\pi/2l)^2|}{R[\lambda_n^2+(m\pi/2l)^2]^2} \leqslant$$
$$\leqslant \frac{2l}{R}\frac{8}{\pi}\frac{[(2/\pi)\lambda_n l][\sigma[(2/\pi)\lambda_n l]^2+(1-\sigma)m^2]}{\{[(2/\pi)\lambda_n l]^2+m^2\}^2} \tag{2.21}$$

Furthermore, substituting the series (2.2) into the second equation of (2.13), we obtain

$$T_n^{(0)} = \frac{2\sigma^2}{(1-2\sigma)\lambda_n lL_n^{(1)}} \qquad (n=1, 2, \ldots) \tag{2.22}$$

We have the identities

$$\frac{8}{\pi}\sum_{m=2,4,\ldots}^{\infty} \frac{km^2}{(k^2+m^2)^2} = \coth\frac{\pi k}{2} - \frac{\pi k/2}{\sinh^2(\pi k/2)}$$

$$\frac{8}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{k^3}{(k^2+m^2)^2}=\tanh\frac{\pi k}{2}-\frac{\pi k/2}{\cosh^2(\pi k/2)} \tag{2.23}$$

$$\frac{8}{\pi}\sum_{m=2,4,\ldots}^{\infty}\frac{k^3}{(k^2+m^2)^2}=\coth\frac{\pi k}{2}+\frac{\pi k/2}{\sinh^2(\pi k/2)}-\frac{4}{\pi k}$$

$$\frac{8}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{km^2}{(k^2+m^2)^2}=\tan\frac{\pi k}{2}+\frac{\pi k/2}{\cosh^2(\pi k/2)}$$

The quantity $T_n^{(1)}$, by virtue of (2.19), (2.21) and (2.23), may be bounded in the following manner:

$$T_n^{(1)}\leqslant\frac{(1-2\theta_1)}{L_n^{(1)}}\left\{(1-\sigma)\left[\coth\lambda_n l-\frac{\lambda_n l}{\sinh^2\lambda_n l}\right]+\sigma\left[\coth\lambda_n l+\frac{\lambda_n l}{\sinh^2\lambda_n l}-\frac{2}{\lambda_n l}\right]\right\}+$$

$$+T_n^{(0)}=\frac{1-2\theta_1}{L_n^{(1)}}\left\{\coth\lambda_n l-(1-2\sigma)\frac{\lambda_n l}{\sinh^2\lambda_n l}-\frac{2\sigma}{\lambda_n l}\right\}+\frac{2\sigma^2}{(1-2\sigma)\lambda_n l L_n^{(1)}}<$$

$$<1-2\theta_1-\frac{2\sigma}{\lambda_n l L_n^{(1)}}\left[1-2\theta_1-\frac{\sigma}{1-2\sigma}\right]$$

where we have used (1.19) and (2.22). From Formula (2.17) it follows that

$$1-2\theta_1-\frac{\sigma}{1-2\sigma}\geqslant 0$$

Therefore

$$T_n^{(1)}<1-2\theta_1 \qquad (n=1,\ 2,\ \ldots) \tag{2.24}$$

Inequalities (2.18) and (2.24) show that the infinite system (1.30) is fully regular for values of σ in the range $0 < \sigma < 1/3$. Further, using (2.20), (2.21) and (2.23) we bound $T_n^{(2)}$

$$T_n^{(2)}\leqslant\frac{1-2\theta_1}{L_n^{(2)}}\left\{\sigma\left[\tanh\lambda_n l-\frac{\lambda_n l}{\cosh^2\lambda_n l}\right]+(1-\sigma)\left[\tanh\lambda_n l+\frac{\lambda_n l}{\cosh^2\lambda_n l}\right]\right\}=$$

$$=\frac{(1-2\theta_1)}{L_n^{(2)}}\left[\tanh\lambda_n l+\frac{(1-2\sigma)\lambda_n l}{\cosh^2\lambda_n l}\right]$$

Hence, applying (1.19), we find

$$T_n^{(2)}<1-\theta_1-\left[\theta_1-2(1-2\theta_1)(1-\sigma)\frac{t_n}{1-t_n}\right],\quad t_n=\frac{2\lambda_n l}{\sinh 2\lambda_n l}\quad (n=1,2,\ldots) \tag{2.25}$$

It is seen from inequalities (2.18) and (2.25) that for arbitrary σ

in the interval $0 < \sigma < 1/3$ and for arbitrary dimensions l and R of the cylinder, a number n_0 can be found such that for all $n > n_0$ the following inequality will be fulfilled

$$\theta_1 - 2(1 - 2\theta_1)(1 - \sigma)\frac{t_n}{1 - t_n} > 0 \qquad (n > n_0)$$

This means that the system (1.31) will be fully quasiregular for all the indicated values of σ.

By the use of Formulas (1.32), (1.23) and (1.29), it is easy to show that the free terms of the infinite systems (1.30) and (1.31) are bounded if the Fourier coefficients of the boundary functions are of the order

$$\psi_m = O(m^{-1}), \quad \chi_m = O(m^{-1}), \quad f_n^{(i)} = O(\sqrt{n}), \quad \varphi_n^{(i)} = O(\sqrt{n}) \quad (i = 1, 2)$$

By the same token, the fully regular infinite system (1.30) has a unique bounded solution. The question of the existence of a unique bounded solution of the fully quasiregular infinite system (1.31) reduces to the existence and uniqueness of the solution of a finite system of n_0 equations in n_0 unknowns. If the solution of this system of finite equations exists and is unique, then a bounded solution of the infinite system (1.31) exists and is unique [17].

3. Displacements prescribed on the ends of the cylinder and stresses prescribed on the lateral surface. It is required to find functions $u(r, z)$ and $w(r, z)$ which satisfy the Lamé differential equations (1.1) in the interior of the cylincer $0 \leqslant r \leqslant R$, $-l \leqslant z \leqslant l$ and which on its surface satisfy the conditions

$$u(r, l) = f_1(r), \quad u(r, -l) = f_2(r), \quad \tau_{rz}(R, z) = 0 \tag{3.1}$$

$$w(r, l) = \varphi_1(r), \quad w(r, -l) = \varphi_2(r), \quad \sigma_r(R, z) = \psi(z) \tag{3.2}$$

Here $\sigma_r(R, z)$ is the normal stress on the lateral surface of the cylinder. The functions $f_1(r)$, $f_2(r)$, $\varphi_1(r)$ and $\varphi_2(r)$ are assumed to be representable in Fourier series

$$\varphi_i(r) = \varphi_0^{(i)} + \sum_{n=1}^{\infty} \varphi_n^{(i)} J_0(\lambda_n r), \quad f_i(r) = \sum_{n=1}^{\infty} f_n^{(i)} J_1(\lambda_n r) \quad \begin{pmatrix} 0 \leqslant r \leqslant R \\ i = 1, 2 \end{pmatrix} \tag{3.3}$$

where $\lambda_n R = \gamma_n$ are positive roots of the equation $J_1(\gamma) = 0$

$$\varphi_0^{(i)} = \frac{2}{R^2}\int_0^R r\varphi_i(r)\,dr, \qquad \varphi_n^{(i)} = \frac{2}{R^2 J_0^2(\lambda_n R)}\int_0^R r\varphi_i(r) J_0(\lambda_n r)\,dr$$

$$f_n^{(i)} = \frac{2}{R^2 J_0^2(\lambda_n R)} \int_0^R r f_i(r) J_1(\lambda_n r)\,dr \qquad (i = 1,\ 2)$$

and the function $\psi(z)$ is assumed to have the Fourier series

$$\psi(z) = \sum_{m=1}^{\infty} \psi_m \sin \frac{m\pi(z-l)}{2l} \qquad (-l \leqslant z \leqslant l) \tag{3.4}$$

We seek a series solution of the boundary value problem in the form

$$u = \sum_{m=1}^{\infty} u_m^{(3)} + \sum_{n=1}^{\infty} u_n^{(4)}, \quad w = a_1 + \frac{1-2\sigma}{2(1-\sigma)} a_3 z + \sum_{m=1}^{\infty} w_m^{(3)} + \sum_{n=1}^{\infty} w_n^{(4)} \tag{3.5}$$

Here

$$u_m^{(3)} = -\left\{\frac{1}{2} C_m^{(1)} I_1(k_m r) + C_m[(4\sigma - 2) I_1(k_m r) + k_m r I_0 k_m r)]\right\} \times \\ \times \frac{\sin[m\pi(z-l)/2l]}{2(1-\sigma) k_m I_1(k_m R)} \tag{3.6}$$

$$w_m^{(3)} = -\left\{\frac{1}{2} C_m^{(1)} I_0(k_m r) + C_m[2 I_0(k_m r) + k_m r I_1(k_m r)]\right\} \frac{\cos[m\pi(z-l)/2l]}{2(1-\sigma) k_m I_1(k_m R)}$$

$$u_n^{(4)} = \frac{1}{4(1-\sigma)\sinh\lambda_n l J_0(\lambda_n R)} \{A_n^{(3)} \sinh\lambda_n z + A_n^{(4)} \tanh\lambda_n l \cosh\lambda_n z + \\ + A_n^{(1)} \tanh\lambda_n l\, z \sinh\lambda_n z + A_n^{(2)} z \cosh\lambda_n z\} J_1(\lambda_n r)$$

$$w_n^{(4)} = \frac{1}{4(1-\sigma)\sinh\lambda_n l J_0(\lambda_n R)} \Big\{- A_n^{(3)} \cosh\lambda_n z - A_n^{(4)} \tanh\lambda_n l \sinh\lambda_n z - \\ - A_n^{(1)} \tanh\lambda_n l\, z \cosh\lambda_n z - A_n^{(2)} z \sinh\lambda_n z + \\ + \frac{(3-4\sigma)}{\lambda_n} [A_n^{(1)} \tanh\lambda_n l \sinh\lambda_n z + A_n^{(2)} \cosh\lambda_n z]\Big\} J_0(\lambda_n r)$$

$$k_m = \frac{m\pi}{2l}, \qquad \lambda_n R = \gamma_n, \qquad J_1(\gamma_n) = 0 \tag{3.7}$$

where a_1, a_3, C_m, $C_m^{(1)}$, $A_n^{(1)}$, $A_n^{(2)}$, $A_n^{(3)}$ and $A_n^{(4)}$ are unknown constants.

The functions (3.6) are obtained by means of Formulas (1.8). Therefore the functions given by the series (3.5) satisfy the differential equations (1.1). The satisfaction of the boundary conditions (3.1), with (3.3) taken into account, leads to the following relations for the unknown constants:

$$C_m^{(1)} = -C_m\left[4\sigma + 2k_m R \frac{I_0(k_m R)}{I_1(k_m R)}\right]$$

$$A_n^{(4)} = -A_n^{(1)} l \tanh \lambda_n l + 2(1-\sigma)[f_n^{(1)} + f_n^{(2)}] J_0(\lambda_n R) \tag{3.8}$$

$$A_n^{(3)} = -A_n^{(2)} l \coth \lambda_n l + 2(1-\sigma)[f_n^{(1)} - f_n^{(2)}] J_0(\lambda_n R)$$

Further, by satisfying the boundary conditions (3.2), we obtain three equations. We then compare Fourier coefficients of the functions entering into these equations, taking into account expansions (3.3), (3.4), and the expansion of unity

$$1 = \sum_{m=1}^{\infty} \frac{2[(-1)^m - 1]}{m\pi} \sin \frac{m\pi(z-l)}{2l} \qquad (-l < z < l)$$

As a result of calculations analogous to those carried out in Section 1, we obtain relationships for the determination of the unknown constants, which, after some transformations, take on the form

$$a_1 = -\frac{2\sigma}{R(1-\sigma)} \sum_{m=2,4,\ldots}^{\infty} \frac{C_m}{k_m^2} + \frac{1}{2}[\varphi_0^{(1)} + \varphi_0^{(2)}] \tag{3.9}$$

$$a_3 = -\frac{4\sigma}{Rl(1-2\sigma)} \sum_{m=1,3,\ldots}^{\infty} \frac{C_m}{k_m^2} + \frac{1-\sigma}{l(1-2\sigma)}[\varphi_0^{(1)} - \varphi_0^{(2)}] \tag{3.10}$$

$$C_m = -\frac{1}{L_m} \sum_{n=1}^{\infty} A_n^{(1)} H_{mn}^{(1)} + \frac{4\sigma a_3}{m\pi L_m} + \xi_m^{(1)} \qquad (m = 1, 3, \ldots) \tag{3.11}$$

$$C_m = -\frac{1}{L_m} \sum_{n=1}^{\infty} A_n^{(2)} H_{mn}^{(1)} + \xi_m^{(2)} \qquad (m = 2, 4, \ldots) \tag{3.12}$$

$$A_n^{(1)} = \frac{2}{L_n^{(1)}} \sum_{m=1,3,\ldots}^{\infty} C_m H_{mn}^{(2)} + \eta_n^{(1)} \qquad (n = 1, 2, \ldots) \tag{3.13}$$

$$A_n^{(2)} = \frac{2}{L_n^{(2)}} \sum_{m=2,4,\ldots}^{\infty} C_m H_{mn}^{(2)} + \eta_n^{(2)} \qquad (n = 1, 2, \ldots) \tag{3.14}$$

Here

$$L_m = k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] - \frac{2-2\sigma}{k_m R} \tag{3.15}$$

$$L_n^{(1)} = (3-4\sigma)\tanh \lambda_n l - \frac{\lambda_n l}{\cosh^2 \lambda_n l}, \qquad L_n^{(2)} = (3-4\sigma)\coth \lambda_n l + \frac{\lambda_n l}{\sinh^2 \lambda_n l} \tag{3.16}$$

$$H_{mn}^{(1)} = \frac{2k_m\,[\lambda_n^2 - \sigma(\lambda_n^2 + k_m^2)]}{l(\lambda_n^2 + k_m^2)^2}, \qquad H_{mn}^{(2)} = \frac{4\lambda_n\,[\lambda_n^2 - \sigma(\lambda_n^2 + k_m^2)]}{R(\lambda_n^2 + k_m^2)^2} \quad (3.17)$$

$$\xi_m^{(1)} = \frac{2(1-\sigma)}{L_m}\sum_{n=1}^{\infty}\frac{\lambda_n k_m\,[f_n^{(1)} + f_n^{(2)}]\,J_0(\lambda_n R)}{l(\lambda_n^2 + k_m^2)} + \frac{1-\sigma}{L_m G}\psi_m \quad (m = 1, 3, \ldots) \quad (3.18)$$

$$\xi_m^{(2)} = \frac{2(1-\sigma)}{L_m}\sum_{n=1}^{\infty}\frac{\lambda_n k_m\,[f_n^{(1)} - f_n^{(2)}]\,J_0(\lambda_n R)}{l(\lambda_n^2 + k_m^2)} + \frac{1-\sigma}{L_m G}\psi_m \quad (m = 2, 4, \ldots) \quad (3.19)$$

$$\eta_n^{(1)} = \frac{2(1-\sigma)\lambda_n J_0(\lambda_n R)}{L_n^{(1)}}\,[\varphi_n^{(1)} - \varphi_n^{(2)} + (f_n^{(1)} + f_n^{(2)})\tanh\lambda_n l] \quad (3.20)$$

$$\eta_n^{(2)} = \frac{2(1-\sigma)\lambda_n J_0(\lambda_n R)}{L_n^{(2)}}\,[\varphi_n^{(1)} + \varphi_n^{(2)} + (f_n^{(1)} - f_n^{(2)})\coth\lambda_n l] \quad (3.21)$$

From Equations (3.10) to (3.14) we obtain two infinite systems of linear algebraic equations for the unknown constants C_m

$$C_m = -\frac{2}{L_m}\sum_{n=1}^{\infty}\sum_{s=1,3,\ldots}^{\infty}\frac{1}{L_n^{(1)}}H_{mn}^{(1)}H_{sn}^{(2)}C_s - \frac{16\sigma^2}{Rl(1-2\sigma)m\pi L_m}\sum_{s=1,3,\ldots}^{\infty}\frac{C_s}{k_s^2} + \delta_m^{(1)}$$
$$(m = 1, 3, \ldots) \quad (3.22)$$

$$C_m = -\frac{2}{L_m}\sum_{n=1}^{\infty}\sum_{s=2,4,\ldots}^{\infty}\frac{1}{L_n^{(2)}}H_{mn}^{(1)}H_{sn}^{(2)}C_s + \delta_m^{(2)} \quad (m=2, 4, \ldots) \quad (3.23)$$

where

$$\delta_m^{(1)} = -\frac{1}{L_m}\sum_{n=1}^{\infty}H_{mn}^{(1)}\eta_n^{(1)} + \xi_m^{(1)} + \frac{4\sigma}{lL_m m\pi}\frac{1-\sigma}{1-2\sigma}[\varphi_0^{(1)} - \varphi_0^{(2)}]$$
$$\delta_m^{(2)} = -\frac{1}{L_m}\sum_{n=1}^{\infty}H_{mn}^{(1)}\eta_n^{(2)} + \xi_m^{(2)} \quad (3.24)$$

By means of (3.8), (3.9), (3.10), (3.13) and (3.14), the constants a_1, a_3, $A_n^{(1)}$, $A_n^{(2)}$, $A_n^{(3)}$, $A_n^{(4)}$, $C_m^{(1)}$, which enter into the series (3.5), are uniquely expressed in terms of the constants C_m and the Fourier coefficients of the boundary functions. The constants C_m are in turn to be found from the infinite systems (3.22) and (3.23). If there exists a unique bounded solution of the infinite systems, then the solution which is represented by the series (3.5) will be unique and within the cylincer $-l \leqslant z \leqslant l$, $0 \leqslant r \leqslant R$ the series (3.5) will converge uniformly and admit of double, termwise differentiation.

We next examine the infinite systems. To do this, we bound the sums of the moduli of the coefficients of the systems (3.22) and (3.23) from above, denoting them by $T_m^{(1)}$ and $T_m^{(2)}$ respectively. It is seen that

$$T_m^{(1)} \leqslant \frac{2}{L_m} \sum_{n=1}^{\infty} |H_{mn}^{(1)}| \Gamma_n^{(1)} + T_m^{(0)} \qquad (m = 1, 3, \ldots) \tag{3.25}$$

$$T_m^{(2)} \leqslant \frac{2}{L_m} \sum_{n=1}^{\infty} |H_{mn}^{(1)}| \Gamma_n^{(2)} \qquad (m = 2, 4, \ldots)$$

where

$$T_m^{(0)} = \frac{16\sigma^2}{Rl(1-2\sigma) m\pi L_m} \sum_{s=1,3,\ldots}^{\infty} \frac{1}{k_s^2} \tag{3.26}$$

$$\Gamma_n^{(1)} = \frac{1}{L_n^{(1)}} \sum_{s=1,3,\ldots}^{\infty} |H_{sn}^{(2)}|, \qquad \Gamma_n^{(2)} = \frac{1}{L_n^{(2)}} \sum_{s=2,4,\ldots}^{\infty} |H_{sn}^{(2)}| \tag{3.27}$$

Here

$$L_m > 0, \qquad L_n^{(1)} > 0, \qquad L_n^{(2)} > 0$$

which follows from Formulas (3.15), (3.16), the identity (2.7), and the inequality $0 < \sigma \leqslant 0.5$. Using the inequality

$$|H_{sn}^{(2)}| = \frac{4\lambda_n |\lambda_n^2 - \sigma[(s\pi/2l)^2 + \lambda_n^2]|}{R[\lambda_n^2 + (s\pi/2l)^2]^2} \leqslant \frac{l}{R}\frac{8}{\pi} \frac{[(2/\pi)\lambda_n l][(1-\sigma)(2\lambda_n l/\pi)^2 + \sigma s^2]}{[(2\lambda_n l/\pi)^2 + s^2]^2} \tag{3.28}$$

the identity (2.23) and the expression (3.16) for $L_n^{(2)}$, we bound $\Gamma_n^{(2)}$

$$\Gamma_n^{(2)} \leqslant \frac{l}{RL_n^{(2)}}\left[(1-\sigma)\left(\coth\lambda_n l + \frac{\lambda_n l}{\sinh^2\lambda_n l} - \frac{2}{\lambda_n l}\right) + \sigma\left(\coth\lambda_n l - \frac{\lambda_n l}{\sinh^2\lambda_n l}\right)\right] =$$

$$= \frac{l}{RL_n^{(2)}}\left[\coth\lambda_n l - \frac{\lambda_n l}{\sinh^2\lambda_n l} - \frac{2-2\sigma}{\lambda_n l}\left\{1 - \left(\frac{\lambda_n l}{\sinh\lambda_n l}\right)^2\right\}\right] < \frac{l\coth\lambda_n l}{RL_n^{(2)}}$$

or

$$\Gamma_n^{(2)} < \frac{l\coth\lambda_n l}{RL_n^{(2)}} < \frac{l}{(3-4\sigma)R} \qquad (n = 1, 2, \ldots) \tag{3.29}$$

In exactly the same way, using (3.28), (2.23) and (3.16), we bound $\Gamma_n^{(1)}$

$$\Gamma_n^{(1)} \leqslant \frac{l}{RL_n^{(1)}}\left[(1-\sigma)\left(\tanh\lambda_n l - \frac{\lambda_n l}{\cosh^2\lambda_n l}\right) + \sigma\left(\tanh\lambda_n l + \frac{\lambda_n l}{\cosh^2\lambda_n l}\right)\right] =$$

$$= \frac{l}{R}\frac{\tanh\lambda_n l + (2\sigma-1)\lambda_n l\,\mathrm{sech}^2\lambda_n l}{(3-4\sigma)\tanh\lambda_n l - \lambda_n l\,\mathrm{sech}^2\lambda_n l} = \frac{l}{R}\frac{1-(1-2\sigma)2\lambda_n l\,\mathrm{cosech}\,2\lambda_n l}{3-4\sigma-2\lambda_n l\,\mathrm{cosech}\,2\lambda_n l}$$

or

$$\Gamma_n^{(1)} \leqslant \frac{l}{R} f(t_n, \sigma) \qquad (n = 1, 2, \ldots)$$

where

$$f(t_n, \sigma) = \frac{1-(1-2\sigma)t_n}{3-4\sigma-t_n}, \qquad t_n = \frac{2\lambda_n l}{\sinh 2\lambda_n l} \qquad (0 < t_n < 1)$$

In [18], the following bound for the function $f(t_n, \sigma)$ was found

$$f(t_n, \sigma) \leqslant 1 - \theta_0 \qquad (n = 1, 2, \ldots)$$

$$\theta_0 = \begin{cases} 2(1-\sigma)/(3-4\sigma) & (0 < \sigma \leqslant 1/4) \\ (1-3\sigma)/(1-2\sigma) & (1/4 < \sigma < 1/2) \end{cases}$$

where

$$\theta_0 > 0 \qquad \text{for } 0 < \sigma < 1/3 \tag{3.30}$$

Therefore

$$\Gamma_n^{(1)} \leqslant \frac{l}{R}(1-\theta_0) \qquad (n = 1, 2, \ldots) \tag{3.31}$$

In view of (3.29) and (3.31), the bounds (3.25) can be rewritten as

$$\begin{aligned} &T_m^{(1)} \leqslant \frac{2l}{R}(1-\theta_0)T_m + T_m^{(0)} \qquad (m = 1, 3, \ldots) \\ &T_m^{(2)} < \frac{2l}{R(3-4\sigma)} T_m \qquad (m = 2, 4, \ldots) \end{aligned} \qquad \left(T_m = \frac{1}{L_m}\sum_{n=1}^{\infty} |H_{mn}^{(1)}|\right) \tag{3.32}$$

Using (3.17) and the identities (2.7), (2.8), we bound T_m

$$T_m = \sum_{n=1}^{\infty} \frac{2k_m|\lambda_n^2 - \sigma(\lambda_n^2 + k_m^2)|}{L_m l(k_m^2+\lambda_n^2)^2} \leqslant \frac{R}{2lL_m}\frac{4}{R}\sum_{n=1}^{\infty}\frac{k_m[(1-\sigma)\lambda_n^2 + \sigma k_m^2]}{(k_m^2+\lambda_n^2)^2} =$$

$$= \frac{R}{2lL_m}\left\{(1-\sigma)\left[2\frac{I_0(k_m R)}{I_1(k_m R)} - k_m R\left(\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right)\right] + \right.$$

$$\left. + \sigma\left[k_m R\left(\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right) - \frac{4}{k_m R}\right]\right\}$$

$$T_m = \frac{R}{2lL_m}\left\{(2-2\sigma)\frac{I_0(k_m R)}{I_1(k_m R)} - (1-2\sigma)k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] - \frac{4\sigma}{k_m R}\right\}$$

Thus, applying inequality (2.11) and Formula (3.15), we obtain

$$T_m < \frac{R}{2lL_m}\left\{(2-2\sigma)k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] + \frac{4-4\sigma}{k_m R} - \right.$$

$$\left. - (1-2\sigma)k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] - \frac{4\sigma}{k_m R}\right\} =$$

$$= \frac{R}{2lL_m}\left\{k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] - \frac{2-2\sigma}{k_m R} + \frac{6-10\sigma}{k_m R}\right\} =$$
$$= \frac{R}{2lL_m}\left\{L_m + \frac{6-10\sigma}{k_m R}\right\} = \frac{R}{2l}\left[1 + \frac{6-10\sigma}{k_m R\, L_m}\right] \qquad (3.33)$$

Since

$$\sum_{s=1,3,\dots}^{\infty} \frac{1}{s^2} = \frac{\pi^2}{8}$$

we find, by substituting $k_s = s\pi/2l$ into (3.26), that

$$T_m^{(0)} = \frac{8\sigma^2 l}{R(1-2\sigma)\, m\pi L_m} = \frac{4\sigma^2}{(1-2\sigma)\, k_m R L_m} \qquad (3.34)$$

Hence, by virtue of (3.33) and (3.34), the bounds (3.32) take on the form

$$T_m^{(1)} < 1 - \theta_0 + \frac{6-10\sigma}{k_m R L_m}(1-\theta_0) + \frac{4\sigma^2}{(1-2\sigma)\, k_m R L_m} \qquad (m = 1, 3, \dots)$$
$$T_m^{(2)} < \frac{1}{3-4\sigma} + \frac{6-10\sigma}{(3-4\sigma)\, k_m R L_m} \qquad (m = 2, 4, \dots)$$

or

$$T_m^{(1)} < 1 - \frac{1}{2}\theta_0 - \left\{\frac{1}{2}\theta_0 - \frac{2}{k_m R L_m}\left[(3-5\sigma)(1-\theta_0) + \frac{2\sigma^2}{1-2\sigma}\right]\right\} \qquad (m = 1, 3, \dots)$$
$$T_m^{(2)} < 1 - \frac{1-2\sigma}{3-4\sigma} - \left[\frac{1-2\sigma}{3-4\sigma} - \frac{2(3-5\sigma)}{(3-4\sigma)\, k_m R L_m}\right] \qquad (m = 2, 4, \dots) \qquad (3.35)$$

It is seen from inequalities (3.30) and (3.35) that for an arbitrary value σ in the interval $0 < \sigma < 1/3$ and for arbitrary dimensions l and R of the cylinder, a number m_0 can be found such that for all $m > m_0$ we will have

$$\frac{1}{2}\theta_0 - \frac{2}{k_m R L_m}\left[(3-5\sigma)(1-\theta_0) + \frac{2\sigma^2}{1-2\sigma}\right] > 0 \qquad (m > m_0)$$
$$\frac{1-2\sigma}{3-4\sigma} - \frac{2(3-5\sigma)}{(3-4\sigma)\, L_m k_m R} > 0 \qquad (m > m_0)$$

It is furthermore obvious that the second inequality can be satisfied for arbitrary σ in the interval $0 < \sigma < 1/2$. This means that for $0 < \sigma < 1/3$ the infinite system (3.22) is fully quasiregular [17], whereas the infinite system (3.23) is fully quasiregular for $0 < \sigma < 1/2$.

It is easy to show that the free terms (3.24) of the infinite systems (3.22) and (3.23) are bounded if the Fourier coefficients of the boundary functions are of the order

$$\psi_m = O(1), \qquad f_n^{(i)} = O(n^{-1/2}), \qquad \varphi_n^{(i)} = O(n^{-1/2}) \qquad (i = 1, 2)$$

Therefore the question of the existence of a unique solution of each of the infinite systems for the indicated values of σ reduces to the existence and uniqueness of a finite system of m_0 equations in m_0 unknowns [17].

4. Displacements prescribed on the surface of the cylinder. The boundary conditions of the problem can be written as

$$w(r, l) = f_1(r), \qquad w(r, -l) = f_2(r), \qquad u(R, z) = \psi(z)$$
$$u(r, l) = \varphi_1(r), \qquad u(r, -l) = \varphi_2(r), \qquad w(R, z) = \chi(z)$$

Here

$$f_i(r) = f_0^{(i)} + \sum_{n=1}^{\infty} f_n^{(i)} J_0(\lambda_n r), \qquad \varphi_i(r) = \sum_{n=1}^{\infty} \varphi_n^{(i)} J_1(\lambda_n r) \qquad \binom{i = 1, 2}{0 \leqslant r \leqslant R}$$

$$\psi(z) = \frac{\psi_0}{2} + \sum_{m=1}^{\infty} \psi_m \cos\frac{m\pi(z-l)}{2l}, \quad \chi(z) = \sum_{m=1}^{\infty} \chi_m \sin\frac{m\pi(z-l)}{2l} \qquad (-l \leqslant z \leqslant l)$$

where $\lambda_n R = \gamma_n$ are the positive roots of the equation $J_1(\gamma) = 0$.

We seek a solution of the boundary value problem in series form

$$\begin{aligned} u &= \frac{a_2 r}{8(1-\sigma)} + l \sum_{m=1}^{\infty} \frac{I_0(k_m R) k_m}{I_1(k_m R)} u_m^{(1)} + \sum_{n=1}^{\infty} u_n^{(5)} \\ w &= a_1 + \frac{z}{2(1-\sigma)}\left[(1-2\sigma) a_3 - \frac{1}{2} a_2\right] + l \sum_{m=1}^{\infty} \frac{k_m I_0(k_m R)}{I_1(k_m R)} w_m^{(1)} + \sum_{n=1}^{\infty} w_n^{(5)} \end{aligned} \tag{4.1}$$

Here $w_m^{(1)}$ and $u_m^{(1)}$ are given by Formulas (1.9)

$$u_n^{(5)} = \frac{\lambda_n R}{4(1-\sigma)\sinh\lambda_n l} [A_n^{(3)} \tanh\lambda_n l \sinh\lambda_n z + A_n^{(4)} \cosh\lambda_n z + A_n^{(1)} z \sinh\lambda_n z + A_n^{(2)} \tanh\lambda_n l \, z \cosh\lambda_n z] J_1(\lambda_n r)$$

$$w_n^{(5)} = \frac{\lambda_n R}{4(1-\sigma)\sinh\lambda_n l} \Big\{- A_n^{(3)} \tanh\lambda_n l \cosh\lambda_n z - A_n^{(4)} \sinh\lambda_n z - A_n^{(1)} z \cosh\lambda_n z - A_n^{(2)} \tanh\lambda_n l \, z \sinh\lambda_n z + \frac{3-4\sigma}{\lambda_n} [A_n^{(1)} \sinh\lambda_n z + A_n^{(2)} \tanh\lambda_n l \cosh\lambda_n z]\Big\} J_0(\lambda_n r)$$

$$\lambda_n R = \gamma_n, \qquad J_1(\gamma_n) = 0$$

where $u_n^{(5)}$ and $w_n^{(5)}$ are particular solutions of Equations (1.1).

The boundary conditions are satisfied by the method set forth in Section 1, and by the use of the expansions

$$r = -\sum_{n=1}^{\infty} \frac{2}{\lambda_n J_0(\lambda_n R)} J_1(\lambda_n r) \qquad (0 \leqslant r < R)$$

$$b_1 + b_2 z = \sum_{m=1}^{\infty} \{b_1[(-1)^m - 1] - lb_2[1 + (-1)^m]\} \frac{2}{m\pi} \sin\frac{m\pi(z-l)}{2l} \qquad (-l < z < l)$$

we obtain the following relationships for the unknown constants entering into the series (4.1):

$$a_1 = \frac{1}{2}[f_0^{(1)} + f_0^{(2)}]$$

$$a_2 = \frac{4}{R}(1-\sigma)\psi_0, \quad (1-2\sigma)a_3 = \frac{1}{2}a_2 + \frac{1-\sigma}{l}[f_0^{(1)} - f_0^{(2)}]$$

$$C_m^{(1)} = -2C_m\left[4\sigma - 2 + k_m R\frac{I_0(k_m R)}{I_1(k_m R)}\right] - \frac{4(1-\sigma)}{l}\psi_m$$

$$A_n^{(3)} = \frac{A_n^{(2)}}{\lambda_n}[3 - 4\sigma - \lambda_n l \tanh \lambda_n l] - \frac{2(1-\sigma)}{\lambda_n R}[f_n^{(1)} + f_n^{(2)}]$$

$$A_n^{(4)} = \frac{A_n^{(1)}}{\lambda_n}[3 - 4\sigma - \lambda_n l \coth \lambda_n l] - \frac{2(1-\sigma)}{\lambda_n R}[f_n^{(1)} - f_n^{(2)}]$$

$$A_n^{(1)} = \frac{4l}{RL_n^{(1)}} \sum_{m=2,4,\ldots}^{\infty} \frac{H_{mn}^{(1)} C_m}{J_0(\lambda_n R)} + \delta_n^{(1)} \qquad (n = 1, 2, \ldots) \tag{4.2}$$

$$A_n^{(2)} = \frac{4l}{RL_n^{(2)}} \sum_{m=1,3,\ldots}^{\infty} \frac{H_{mn}^{(1)} C_m}{J_0(\lambda_n R)} + \delta_n^{(2)} \qquad (n = 1, 2, \ldots) \tag{4.3}$$

$$C_m = \frac{2R}{lL_m} \sum_{n=1}^{\infty} J_0(\lambda_n R) H_{mn}^{(2)} A_n^{(1)} + \beta_m^{(1)} \qquad (m = 2, 4, \ldots) \tag{4.4}$$

$$C_m = \frac{2R}{lL_m} \sum_{n=1}^{\infty} J_0(\lambda_n R) H_{mn}^{(2)} A_n^{(2)} + \beta_m^{(2)} \qquad (m = 1, 3, \ldots) \tag{4.5}$$

$$\delta_n^{(1)} = \frac{2(1-\sigma)}{RL_n^{(1)}} \left\{ \sum_{m=2,4,\ldots}^{\infty} \frac{4\lambda_n \psi_m}{RJ_0(\lambda_n R)(\lambda_n^2 + k_m^2)} + \varphi_n^{(1)} + \varphi_n^{(2)} + \right.$$
$$\left. + \frac{2\psi_0}{R\lambda_n J_0(\lambda_n R)} + [f_n^{(1)} - f_n^{(2)}] \coth \lambda_n l \right\}$$

$$\delta_n^{(2)} = \frac{2(1-\sigma)}{RL_n^{(2)}} \left\{ \sum_{m=1,3,\ldots}^{\infty} \frac{4\lambda_n \psi_m}{RJ_0(\lambda_n R)(\lambda_n^2 + k_m^2)} + \right.$$
$$\left. + \varphi_n^{(1)} - \varphi_n^{(2)} + [f_n^{(1)} + f_n^{(2)}] \tanh \lambda_n l \right\}$$

$$\beta_m^{(1)} = \frac{2(1-\sigma)}{lL_m} \left\{ \frac{\psi_m I_0(k_m R)}{I_1(k_m R)} + \sum_{n=1}^{\infty} \frac{[f_n^{(1)} - f_n^{(2)}] J_0(\lambda_n R) k_m}{l(\lambda_n^2 + k_m^2)} + \chi_m + \right.$$
$$\left. + \frac{2l[(1-2\sigma)a_3 - a_2/2]}{(1-\sigma)m\pi} \right\}$$

$$\beta_m^{(2)} = \frac{2(1-\sigma)}{lL_m}\left\{\frac{\psi_m I_0(k_m R)}{I_1(k_m R)} + \sum_{n=1}^{\infty}\frac{[f_n^{(1)}+f_n^{(2)}]J_0(\lambda_n R)k_m}{l(k_m^2+\lambda_n^2)} + \chi_m + \frac{4\sigma_1}{m\pi}\right\}$$

$$L_n^{(1)} = (3-4\sigma)\coth\lambda_n l - \frac{\lambda_n l}{\sinh^2\lambda_n l}, \qquad L_n^{(2)} = (3-4\sigma)\tanh\lambda_n l + \frac{\lambda_n l}{\cosh^2\lambda_n l} \tag{4.6}$$

$$L_m = (4-4\sigma)\frac{I_0(k_m R)}{I_1(k_m R)} - k_m R\left[\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right] \tag{4.7}$$

$$H_{mn}^{(1)} = \frac{2\lambda_n k_m^2}{R(\lambda_n^2+k_m^2)^2}, \qquad H_{mn}^{(2)} = \frac{\lambda_n^2 k_m}{l(k_m^2+\lambda_n^2)^2} \tag{4.8}$$

We introduce new unknown constants X_n and Y_n by setting

$$A_n^{(1)} = \frac{X_n}{J_0(\lambda_n R)}, \qquad A_n^{(2)} = \frac{Y_n}{J_0(\lambda_n R)} \tag{4.9}$$

We change the subscript n to s in Equations (4.2) and (4.3) and substitute into these equations (4.4), (4.5) and (4.9). Thereby we obtain two infinite systems of linear algebraic equations in the unknowns X_n and Y_n

$$X_s = \frac{8}{L_s^{(1)}}\sum_{m=2,4,\ldots}^{\infty}\sum_{n=1}^{\infty}\frac{1}{L_m}H_{ms}^{(1)}H_{mn}^{(2)}X_n + \gamma_s^{(1)} \qquad (s=1,2,\ldots) \tag{4.10}$$

$$Y_s = \frac{8}{L_s^{(2)}}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1}^{\infty}\frac{1}{L_m}H_{ms}^{(1)}H_{mn}^{(2)}Y_n + \gamma_s^{(2)} \qquad (s=1,2,\ldots) \tag{4.11}$$

where

$$\begin{aligned}\gamma_s^{(1)} &= \frac{4l}{RL_s^{(1)}}\sum_{m=2,4,\ldots}^{\infty}H_{ms}^{(1)}\beta_m^{(1)} + J_0(\lambda_s R)\delta_s^{(1)}\\ \gamma_s^{(2)} &= \frac{4l}{RL_s^{(2)}}\sum_{m=1,3,\ldots}^{\infty}H_{ms}^{(1)}\beta_m^{(2)} + J_0(\lambda_s R)\delta_s^{(2)}\end{aligned} \tag{4.12}$$

We next show that these infinite systems are fully regular for $0 < \sigma < 1/2$. We denote the sums of the moduli of the coefficients of the systems (4.10) and (4.11) by $T_s^{(1)}$ and $T_s^{(2)}$ respectively. It is seen that

$$T_s^{(1)} = \frac{8}{L_s^{(1)}}\sum_{m=2,4,\ldots}^{\infty}H_{ms}^{(1)}\Gamma_m, \quad T_s^{(2)} = \frac{8}{L_s^{(2)}}\sum_{m=1,3,\ldots}^{\infty}H_{ms}^{(1)}\Gamma_m \quad (s=1,2,\ldots) \tag{4.13}$$

Here

$$\Gamma_m = \sum_{n=1}^{\infty}\frac{1}{L_m}H_{mn}^{(2)}, \qquad L_s^{(1)} > 0, \qquad L_s^{(2)} > 0, \qquad L_m > 0$$

where the inequalities follow from (4.6), (4.7) and the identity (2.8). Using (4.8), (2.8) and (4.7), we find

$$\Gamma_m = \frac{1}{L_m}\sum_{n=1}^{\infty}\frac{\lambda_n^2 k_m}{l(k_m^2+\lambda_n^2)^2} = \frac{R}{4lL_m}\left[2\frac{I_0(k_m R)}{I_1(k_m R)} - k_m R\left(\frac{I_0^2(k_m R)}{I_1^2(k_m R)} - 1\right)\right] =$$
$$= \frac{R}{4lL_m}\left[L_m - (2-4\sigma)\frac{I_0(k_m R)}{I_1(k_m R)}\right] < \frac{R}{4l}\left[1 - \frac{2-4\sigma}{4-4\sigma}\right] = \frac{R}{4l}\frac{1}{2-2\sigma}$$

Therefore we obtain the following bounds for the quantities (4.13)

$$T_s^{(1)} < \frac{R}{l(1-\sigma)L_s^{(1)}}\sum_{m=2,4,\dots}^{\infty} H_{ms}^{(1)}, \qquad T_s^{(2)} < \frac{R}{l(1-\sigma)L_s^{(2)}}\sum_{m=1,3,\dots}^{\infty} H_{ms}^{(1)}$$
$$(s = 1, 2, \dots) \qquad (4.14)$$

Substituting $H_{ms}^{(1)}$ and $k_m = m\pi/2l$ into (4.14) and using the expressions (2.23) and (4.6) we obtain a bound for $T_s^{(1)}$

$$T_s^{(1)} < \frac{1}{2(1-\sigma)L_s^{(1)}}\left[\coth\lambda_s l - \frac{\lambda_s l}{\sinh^2\lambda_s l}\right] \leqslant \frac{1}{2(1-\sigma)(3-4\sigma)}$$

or

$$T_s^{(1)} < 1 - \theta_2, \quad \theta_2 = 1 - \frac{1}{2(1-\sigma)(3-4\sigma)} \qquad \begin{pmatrix} s = 1, 2, \dots \\ 0 < \sigma \leqslant 1/2 \end{pmatrix} \qquad (4.15)$$

Since $2(1 - \sigma)(3 - 4\sigma) > 1$ for $0 < \sigma < 1/2$, we have

$$\theta_2 > 0, \text{ if } 0 < \sigma < 1/2; \ \theta_2 = 0, \text{ if } \sigma = 1/2 \qquad (4.16)$$

In exactly the same way we find a bound for $T_s^{(2)}$

$$T_s^{(2)} < \frac{1}{2(1-\sigma)L_s^{(2)}}\left[\tanh\lambda_s l + \frac{\lambda_s l}{\cosh^2\lambda_s l}\right] = \frac{1}{2(1-\sigma)} \qquad \frac{1 + 2\lambda_s l \,\mathrm{cosech}\, 2\lambda_s l}{3 - 4\sigma + 2\lambda_s l\,\mathrm{cosech}\,2\lambda_s l}$$

or

$$T_s^{(2)} < \frac{f(t_s, \sigma)}{2(1-\sigma)}, \quad f(t_s, \sigma) = \frac{1+t_s}{3-4\sigma+t_s}, \quad t_s = \frac{2\lambda_s l}{\sinh 2\lambda_s l}, \quad 0 < t_s < 1 \quad (4.17)$$

It is easy to show that

$$f(t_s, \sigma) \leqslant f(1, \sigma) = \frac{1}{2(1-\sigma)} \qquad \begin{pmatrix} 0 < \sigma \leqslant 1/2 \\ 0 \leqslant t_s \leqslant 1 \end{pmatrix}$$

Therefore the bound (4.17) may be rewritten as

$$T_s^{(2)} < 1 - \theta_3; \qquad \theta_3 = 1 - \frac{1}{4(1-\sigma)^2} \qquad \begin{pmatrix} s = 1, 2, \dots \\ 0 < \sigma \leqslant 1/2 \end{pmatrix} \qquad (4.18)$$

$$\theta_3 > 0, \quad \text{if} \quad 0 < \sigma < 1/2; \; \theta_3 = 0, \quad \text{if} \quad \sigma = 1/2$$

From inequalities (4.15) to (4.18) it follows that the infinite systems (4.10) and (4.11) are fully regular for $0 < \sigma < 1/2$ and regular for $\sigma = 1/2$. The free terms (4.12) of the infinite systems (4.10) and (4.11) are bounded if the Fourier coefficients of the boundary functions are of the order

$$\psi_m = O(1), \quad \chi_m = O(1), \quad f_n^{(i)} = O(\sqrt{n}), \quad \varphi_n^{(i)} = O(\sqrt{n}) \quad (i = 1, 2)$$

By the same token, each of the systems (4.10) and (4.11) has a unique bounded solution of $0 < \sigma < 1/2$.

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